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Strong Large Deviation and Local Limit Theorems†

by

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Abstract

Most large deviation results give asymptotic expressions to $\log P(Y_n \geq y_n)$ where the event $\{Y_n \geq y_n\}$ is a large deviation event, that is, $P(Y_n \geq y_n)$ goes to zero exponentially fast. We refer to such results as weak large deviation results. In this paper we obtain strong large deviation results for arbitrary random variables $\{Y_n\}$, that is, we obtain asymptotic expressions for $P(Y_n \geq y_n)$ where $\{Y_n \geq y_n\}$ is a large deviation event. These strong large deviation results are obtained for lattice valued and nonlattice valued random variables and require some conditions on their moment generating functions. These results strengthen existing results which apply mainly to sums of independent and identically distributed random variables.

Let μ be the lebesgue measure on R. Let S be a measurable subset of R such that $0 < \mu(S) < \infty$ and let $b_n \to \infty$. Define $q_n(y;\, b_n,\, S) = [((b_n/\mu(S))\, P(b_n(Y_n - y) \in S)]$, as the pseudo-density function of Y_n. By a local limit theorem we mean the convergence of $q_n(y_n;\, b_n,\, S)$ as $n \to \infty$ and $y_n \to y^*$. In this paper we obtain local limit theorems for arbitrary random variables based on easily verifiable conditions on their characteristic functions. These local limit theorems play a major role in the proofs of the strong large deviation results of this paper.

We illustrate these results with two typical applications.

1. Introduction

The establishment of a limit distribution for a sequence of random variables $\{Y_n, n \geq 1\}$ provides an approximation to $P(Y_n \leq y)$. However, there are other aspects relating to the distribution of Y_n for which one often desires an approximation. This could be $P(Y_n \geq y_n)$, known in the literature as a large deviation, especially when it tends to zero exponentially fast. Another example is $f_n(y_n)$, the probability density function of Y_n at y_n. The term, a large deviation local limit result for Y_n, is used when an asymptotic expression is established for $f_n(y_n)$ and y_n is in the range of a large deviation for Y_n. Still another example is the limit of $(b_n/\mu(S))\, P(b_n(Y_n - y_n) \in S)$, where μ is the lebesgue measure and S is a measurable set such that $0 < \mu(S) < \infty$ and $b_n \to \infty$. Such a result will be referred to as a local limit result for Y_n. This paper will deal with strong large deviation and local limit theorems for arbitrary random variables.

The theory of large deviations for sums of independent and identically distributed (i.i.d.) random variables and its many generalizations has a long history, see for instance Cramér(1938), Chernoff(1952), Ellis(1984), Varadhan(1984) etc. However, most of these results give asymptotic expressions for $\log P(Y_n \geq y_n)$ and so we choose to call them weak large deviation results. For arbitrary random variables T_n and $Y_n = T_n/a_n$ for some sequence $a_n \to \infty$, this paper gives asymptotic expressions for $P(Y_n \geq y_n)$, which we call strong large deviation results. These results are found in Theorems 3.1, 3.5 4.5, 4.7, 4.8, 4.9 and 4.13 which impose conditions on the moment generating function (m g.f.) of T_n. These extend the well-known strong large deviation results for sums of i.i.d. random variables due to Bahadur and Ranga Rao(1960).

The proofs of our strong large deviation theorems depend on the local limit results for Y_n. These are established in this paper in Theorems 2.1, 2.2, 2.3, 4.1 and 4.2 and they are in the spirit of Feller(1967) wherein can be found some of the first local limit results for sums of i.i.d. random variables. Local limit results for extreme values are established

in de Haan and Resnick(1982). Local limit results for sums of triangular arrays of i.i.d. random variables can be found in Jain and Pruitt(1987). The local limit results in this paper apply to arbitrary random variables Y_n and require some easily verifiable conditions on their characteristic functions.

We illustrate our general results with two applications in Section 5. The first application is a local limit result for sums of dependent random variables given by a general model considered in Chaganty and Sethuraman(1987). The second application is a strong large deviation result for the Wilcoxon signed- rank statistic under the null hypothesis.

We do not study large deviation local limit results in this paper. We have obtained such results for arbitrary random variables in Chaganty and Sethuraman(1985) for one-dimensional random variables and in Chaganty and Sethuraman(1986) for multi-dimensional random variables.

2. Local Limit Theorems

Let $\{Y_n, n \geq 1\}$ be a sequence of random variables taking values in R_1, which converge to Y in distribution. Let S be a measurable subset with $0 < \mu(S) < 1$ and let $b_n \to \infty$. Define

$$(2\text{-}1) \qquad q_n(y;\, b_n,\, S) = \frac{b_n}{\mu(S)} P\big(b_n(Y_n - y) \in S\big).$$

Since Y_n may not have a probability density function (p.d.f.), we will consider limiting properties of $q_n(y;\, b_n,\, S)$, which will be called the pseudo-density function of Y_n. Let $\{y_n\}$ be a sequence of real numbers such that $y_n \to y^*$. The convergence of $q_n(y_n;\, b_n,\, S)$ to the p.d.f. of Y at y^* is referred to as a local limit theorem. This is the spirit under which local limit theorems have been studied for normalized sums of i.i.d. random variables by Feller(1967), for normalized extreme values in de Haan and Resnick(1982) and for normalized triangular arrays of i.i.d. random variables in Jain and Pruitt(1985). This section is devoted to local limit theorems for arbitrary random variables Y_n.

To motivate the main Theorems 2.2 and 2.3 of this section we begin with the following result which must be well known.

Theorem 2.1. Let $\{Y_n, n \geq 1\}$ be a sequence of real valued random variables which converge to Y in distribution. Let $\hat{f}_n$ be the characteristic function (c.f.) of Y_n for $n \geq 1$ and let $\hat{f}$ be the c.f. of Y. Suppose that there exists an integrable function $f^*(t)$ such that

$$(2\text{-}2) \qquad \sup_n |\hat{f}_n(t)| \leq f^*(t)$$

for all t. Then Y_n possesses a bounded and continuous p.d.f. f_n and Y also possesses a bounded and continuous p.d.f. f. Let $y_n \to y^*$. Then $f_n(y_n) \to f(y^*)$.

Proof. Condition (2-2) implies that the c.f.'s $\hat{f}_n$ and $\hat{f}$ are integrable. Hence both Y_n and Y possess a bounded and continuous p.d.f.'s. The inversion formula and the dominated convergence theorem show that $f_n(y_n) \to f(y^*)$ if $y_n \to y^*$ as $n \to \infty$.◇

From this theorem we can also conclude that $q_n(y_n; b_n, S) \to f(y^*)$, for all S such that $0 < \mu(S) < \infty$, if $b_n \to \infty$ and $y_n \to y^*$ as $n \to \infty$, where $q_n(y; b_n, S)$ is as defined in (2-1). Even this elementary result finds application in certain situations. For instance, we apply this result in Example 5.1 of Section 5 in this paper. However, condition (2-2) is simply too strong to be useful in most situations. We show in Theorem 2.2 below that by bounding $\hat{f}_n(t)$ on increasing sequences of bounded intervals by an integrable function we can get a result similar to that of Theorem 2.1.

Theorem 2.2. Let $\{Y_n, n \geq 1\}$ be a sequence of random variables which converges to Y in distribution. Let $\hat{f}_n$ be the characteristic function (c.f.) of Y_n for $n \geq 1$ and let $\hat{f}$ be the c.f. of Y. Let $\{b_n\}$ be a sequence of real numbers such that $b_n \to \infty$. Suppose that there exists a sequence $\{\beta_n\}$ with $\beta_n \to \infty$, $\beta_n / b_n \to \infty$ and an integrable function $f^*(t)$ such that

$$\sup_n |\hat{f}_n(t)| \, I(|t| < \beta_n) \leq f^*(t) \tag{2-3}$$

for each t. Then the random variable Y possesses a bounded and continuous p.d.f. f. Let S be a measurable set such that $0 < \mu(S^0) = \mu(\overline{S}) < \infty$. Let $q_n(y; b_n, S)$ be as defined in (2-1). Then there exists a finite constant M and an integer n_s such that

$$\sup_y \left[q_n(y; b_n, S)\right] \leq M \tag{2-4}$$

for $n \geq n_s$. Furthermore, if $y_n \to y^*$ then

$$q_n(y_n; b_n, S) \to f(y^*) \tag{2-5}$$

as $n \to \infty$.

Proof. Since $\hat{f}_n(t) \to \hat{f}(t)$ pointwise and $\beta_n \to \infty$, condition (2-3) implies that $\hat{f}$ is bounded by f^*. Hence Y possesses a bounded and continuous p.d.f. f. Let U_n be the uniform distribution on the set $-S/b_n$ and u_n, $\hat{u}_n$ be the p.d.f. and c.f. corresponding to U_n. We also introduce another distribution function (d.f.) V_n with p.d.f. v_n and c.f. $\hat{v}_n$ as defined below, to obtain the important identity (2-10):

(2-6) $$v_n(x) = \frac{\beta_n}{2\pi}\left[\frac{\sin(\beta_n x/2)}{(\beta_n x/2)}\right]^2, -\infty < x < \infty \qquad \text{and}$$

(2-7) $$\hat{v}_n(t) = \begin{cases} 1 - \frac{|t|}{\beta_n} & \text{if } |t| \le \beta_n \\ 0 & \text{otherwise.} \end{cases}$$

Let F_n be the distribution function (d.f.) of Y_n, and let $Q_n = F_n * U_n$, $M_n = Q_n * V_n$ where $*$ denotes the convolution operation. Notice that $q_n(y; b_n, S)$ defined in (2-1) is the p.d.f. of Q_n. The p.d.f. $m_n(y)$ of M_n is given by

(2-8) $$m_n(y) = \int_{-\infty}^{\infty} q_n(y - x; b_n, S)\, v_n(x)\, dx.$$

Since the c.f. $\hat{m}_n(t)$ of M_n, which is equal to $\hat{f}_n(t)\hat{u}_n(t)\hat{v}_n(t)$, vanishes outside the interval $[-\beta_n, \beta_n]$, the inversion formula gives

(2-9) $$m_n(y) = \frac{1}{2\pi}\int_{-\beta_n}^{\beta_n} \exp(-ity)\hat{m}_n(t)dt.$$

Thus, we get

(2-10) $$\begin{aligned} &\frac{b_n}{\mu(S)}\int_{-\infty}^{\infty} P(b_n(Y_n - y + x) \in S)\, v_n(x)\, dx \\ &= \frac{1}{2\pi}\int_{-\beta_n}^{\beta_n} \exp(-ity_n)\hat{m}_n(t)\, dt \\ &= A_n(y) \quad \text{(say).} \end{aligned}$$

Relation (2-10) is the starting point of the main part of this proof and it relates $q_n(y; b_n, S)$ to the integrable c.f. $\hat{m}_n(t)$. We first show that $A_n(y_n)$ converges to $f(y^*)$ and then obtain lower and upper bounds for $A_n(y_n)$ which depend on $q_n(y_n; b_n, S)$. This will then establish (2-4) and (2-5). By condition (2-3), the dominated convergence theorem and the inversion formula we get that

$$
\begin{aligned}
A_n(y_n) &= \frac{1}{2\pi}\int_{-\beta_n}^{\beta_n} \exp(-ity_n)\,\hat{m}_n(t)\,dt \to \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ity^*)\hat{f}(t)\,dt \\
&= f(y^*)
\end{aligned}
\tag{2-11}
$$

Let $\eta > 0$. Let $s(x, \eta)$ be a closed interval centered at x, that is,

$$
s(x, \eta) = \{y : |y - x| \le \eta\}
\tag{2-12}
$$

and let $S_\eta = \{x : s(x, \eta) \subset S\}$ and $S^\eta = \{y : |y - x| \le \eta, \text{ for some } x \in S\}$. We choose our $\eta (= \eta_s)$ such that $\mu(S_\eta) > 0$ and $[\mu(S^\eta)/\mu(S)] \le 2$. Note that $y \in S_\eta$ implies that $y + x \in s(y, \eta) \subset S$ if $|x| \le \eta$. Therefore, we get a lower bound for $A_n(y)$ as follows:

$$
\begin{aligned}
A_n(y) &\ge \frac{b_n}{\mu(S)} \int_{|x| \le \eta/b_n} P\big(b_n(Y_n - y + x) \in S\big)\, v_n(x)\,dx \\
&\ge \frac{b_n}{\mu(S)} P\big(b_n(Y_n - y) \in S_\eta\big) \int_{|x| \le \eta/b_n} v_n(x)\,dx \\
&\ge \frac{b_n}{\mu(S)} P\big(b_n(Y_n - y) \in S_\eta\big) \left[1 - \frac{4b_n}{\pi\,\beta_n \eta}\right].
\end{aligned}
\tag{2-13}
$$

Combining (2-10) and (2-13) and using condition (2-3) we get

$$
\frac{b_n}{\mu(S)} P\big(b_n(Y_n - y) \in S_\eta\big) \left[1 - \frac{4b_n}{\pi\,\beta_n \eta}\right] \le \frac{1}{\pi} \int_{-\infty}^{\infty} f^*(t)\,dt
\tag{2-14}
$$

for all $n \geq 1$. Replacing S by S^η and noting that $S \subset (S^\eta)_\eta$ we get

$$\frac{b_n}{\mu(S)} P(b_n(Y_n - y) \in S)\left[1 - \frac{4b_n}{\pi\beta_n\eta}\right] \leq \frac{\mu(S^\eta)}{\mu(S)}\frac{1}{\pi}\int_{-\infty}^{\infty} f^*(t)\,dt \leq \frac{2}{\pi}\int_{-\infty}^{\infty} f^*(t)\,dt. \tag{2-15}$$

Since $b_n/\beta_n \to 0$ as $n \to \infty$ we can find an integer n_s so that

$$\sup_y \left[\frac{b_n}{\mu(S)} P(b_n(Y_n - y) \in S)\right] \leq M \tag{2-16}$$

for $n \geq n_s$, where

$$M = \frac{3}{\pi}\int_{-\infty}^{\infty} f^*(t)\,dt. \tag{2-17}$$

This proves assertion (2-4). Note that $y \in S$ implies that $y - x \in S^\eta$ for $|x| \leq \eta$. Therefore for $n \geq n_s$ an upperbound for $A_n(y)$ is given by

$$\begin{aligned} A_n(y) &= \frac{b_n}{\mu(S)} \int_{-\infty}^{\infty} P\big(b_n(Y_n - y + x) \in S\big) v_n(x)\,dx \\ &\leq \frac{b_n}{\mu(S)} P\big(b_n(Y_n - y) \in S^\eta\big) \int_{|x|\leq\eta/b_n} v_n(x)\,dx + M\int_{|x|>\eta/b_n} v_n(x)\,dx \\ &\leq \frac{b_n}{\mu(S)} P\big(b_n(Y_n - y) \in S^\eta\big) + \frac{4Mb_n}{\pi\beta_n\eta}. \end{aligned} \tag{2-18}$$

Thus, from (2-11), (2-13) and (2-18) we get that

$$\begin{aligned} &\limsup_n \frac{b_n}{\mu(S)} P\big(b_n(Y_n - y_n) \in S_\eta\big) \\ &\qquad \leq f(y^*) \leq \liminf_n \frac{b_n}{\mu(S)} P\big(b_n(Y_n - y_n) \in S^\eta\big). \end{aligned} \tag{2-19}$$

Replacing S by S^η in the l.h.s. and S by S_η in the r.h.s. and using the relations $S \subset (S^\eta)_\eta$ and $(S_\eta)^\eta \subset S$ we get that

$$\limsup_n \frac{b_n}{\mu(S^\eta)} P(b_n(Y_n - y_n) \in S) \tag{2-20}$$
$$\leq f(y^*) \leq \liminf_n \frac{b_n}{\mu(S_\eta)} P(b_n(Y_n - y_n) \in S).$$

Letting $\eta \to 0$ and using the fact $\mu(S^0) = \mu(\overline{S})$ we get the assertion (2-5). $\diamond$

Theorem 2.2 was stated as if the sequence $\{b_n\}$ was given first and we were looking for conditions under which conclusions (2-4) and (2-5) hold. One can also see from the proof of this theorem that if $\{\beta_n\}$ is any sequence with $\beta_n \to \infty$ such that (2-3) holds then conclusions (2-4) and (2-5) hold for any sequnce $\{b_n\}$ with $b_n \to \infty$ and $\beta_n/b_n \to \infty$. The next theorem explores the case where the sequence $\{b_n\}$ is such that $b_n \to \infty$ but $\beta_n = O(b_n)$.

Theorem 2.3. Let $\{Y_n, n \geq 1\}$ be a sequence of random variables which converges to Y in distribution. Let $\hat{f}_n$ be the characteristic function (c.f.) of Y_n for $n \geq 1$ and let $\hat{f}$ be the c.f. of Y. Assume that there exists a sequence β_n with $\beta_n \to \infty$ and an integrable function $f^*(t)$ such that (2-3) holds for each t. Let $\{b_n\}$ be a sequence of real numbers such that $b_n \to \infty$ and $\beta_n = O(b_n)$. Assume that

$$\theta_n(\lambda) \stackrel{\text{def}}{=} \sup_{\beta_n \leq |t| \leq \lambda b_n} |\hat{f}_n(t)| = o\left(\frac{1}{b_n}\right) \tag{2-21}$$

for each $\lambda > 0$, as $n \to \infty$. Then the random variable Y possesses a bounded and continuous p.d.f. f. Further the conclusions (2-4) and (2-5) of Theorem 2.2 hold if $y_n \to y^*$ and S be a measurable set such that $0 < \mu(S^0) = \mu(\overline{S}) < \infty$.

Proof. Since $\beta_n \to \infty$, condition (2-3) implies that $\hat{f}$ is bounded by f^*. Hence Y possesses a bounded and continuous p.d.f. f. Also, since $b_n\theta_n(\lambda) \to 0$, for each $\lambda > 0$, we can find a sequence $\{\lambda_n\}$ satisfying

$$\lambda_n \to \infty \text{ and } \lambda_n b_n \theta_n \to 0 \tag{2-22}$$

as $n \to \infty$, where $\theta_n = \theta_n(\lambda_n)$. Change the definitions of the distribution function V_n with p.d.f. v_n and c.f. $\hat{v}_n$ in the proof of Theorem 2.2 as follows:

$$v_n(x) = \frac{\lambda_n b_n}{2\pi}\left[\frac{\sin(\lambda_n b_n x/2)}{(\lambda_n b_n x/2)}\right]^2, -\infty < x < \infty \qquad \text{and} \tag{2-23}$$

$$\hat{v}_n(t) = \begin{cases} 1 - \frac{|t|}{\lambda_n b_n} & \text{if } |t| \le \lambda_n b_n \\ 0 & \text{otherwise.} \end{cases} \tag{2-24}$$

It is easy to see that the same steps in the proof of Theorem 2.2 establish this theorem. $\Diamond$

Remark 2.4. The conclusions of Theorem 2.3 hold if we replace condition (2-21) by

$$\int_{\beta_n \le |t| \le \lambda b_n} |\hat{f}_n(t)|\, dt \to 0 \quad \text{as } n \to \infty, \tag{2-25}$$

for each $\lambda > 0$. Notice that condition (2-21) is needed only to show that $A_n(y_n) \to f(y^*)$, where $A_n(y)$ is as defined in (2-10) with β_n replaced by $\lambda_n b_n$. If (2-25) holds for each $\lambda > 0$ we can find a sequence of real numbers $\{\lambda_n\}$, such that $\lambda_n \to \infty$ and

$$\int_{\beta_n \le |t| \le \lambda_n b_n} |\hat{f}_n(t)|\, dt \to 0 \quad \text{as } n \to \infty, \tag{2-26}$$

from which it follows that $A_n(y_n) \to f(y^*)$.

The next theorem provides a convienent way to verify condition (2-3) of Theorem 2.2. In Lemma 3.3 of Section 3 we will use Theorem 2.3 and this method of verification of condition (2-3).

Theorem 2.5. Let $\{Y_n, n \geq 1\}$ be a sequence of random variables with c.f.'s $\{\hat{f}_n(t)\}$. Let $\{d_n\}$ be a sequence of real numbers such that $d_n \to \infty$. Assume that there exists $\delta > 0$ such that $g_n(t) = d_n^{-2} \log |\hat{f}_n(d_n t)|$ is finite and twice differentiable in the interval $(-\delta, \delta)$, for all $n \geq 1$. Suppose that there exists $\alpha > 0$ such that for $|t| < \delta$,

$$-g_n''(t) \geq \alpha \tag{2-27}$$

for all $n \geq 1$. Then condition (2-3) of Theorem 2.2 is satisfied with $\beta_n = \delta d_n$.

Proof. An application of Taylor's theorem yields for $|t| < \delta$,

$$\begin{aligned} g_n(t) &= g_n(0) + t\, g_n'(0) + \frac{t^2}{2} g_n''(r_n) \\ &= \frac{t^2}{2} g_n''(r_n) \\ &\leq -\frac{\alpha t^2}{2}, \end{aligned} \tag{2-28}$$

where r_n is such that $|r_n| < |t| < \delta$. Therefore for $|t| < \delta d_n$,

$$g_n(t/d_n) \leq -\frac{\alpha t^2}{2d_n^2}. \tag{2-29}$$

Let $\beta_n = \delta d_n$. Thus for $|t| < \beta_n$, we have for all $n \geq 1$,

$$\begin{aligned} |\hat{f}_n(t)| &= \exp(d_n^2(g_n(t/d_n))) \\ &\leq \exp(-\alpha t^2/2), \end{aligned} \tag{2-30}$$

which is an integrable function. This completes the proof of the theorem. ◇

The next lemma obtains a limit for the Laplace transform of Y_n when (2-4) and (2-5) hold. Such a result plays an important role in the proofs of the strong large deviation theorems of Sections 3 and 4.

Lemma 2.6. Let $\{Y_n, n \geq 1\}$ be a sequence of random variables. Let $\{b_n\}$ be a sequence of real numbers such that $b_n \to \infty$. Let S be a measurable set such that $0 < \mu(S) < \infty$. Let $q_n(y; b_n, S)$ be as defined in (2-1). Assume that $q_n(y; b_n, S)$ satisfies (2-4) and (2-5). Then

(2-31)
$$b_n\, E\left[\exp(-b_n Y_n)\, I(Y_n \geq 0)\right] \to f(0)$$

as $n \to \infty$.

Proof. Let $h > 0$. Consider

(2-32)
$$\begin{aligned} I_n &= E\left[\exp(-b_n Y_n)\; I(Y_n \geq 0)\right] \\ &= \sum_{k=1}^{\infty} E\left[\exp(-b_n Y_n)\; I\left(\frac{(k-1)h}{b_n} \leq Y_n < \frac{kh}{b_n}\right)\right] \\ &= \sum_{k=1}^{\infty} E\left[\exp(-b_n Y_n)\; I\left(-\frac{h}{2b_n} \leq Y_n - y_{nk} < \frac{h}{2b_n}\right)\right] \end{aligned}$$

where $y_{nk} = (2k-1)h/b_n$. Let $k_h = [1/h^2]$ and $S_h = [-h/2, h/2)$. We now get lower and upper bounds for I_n as follows:

(2-33)
$$\begin{aligned} I_n &\geq \sum_{k=1}^{k_h} \exp(-kh)\; P\left(-\frac{h}{2b_n} \leq Y_n - y_{nk} < \frac{h}{2b_n}\right) \\ &= \frac{h}{b_n} \sum_{k=1}^{k_h} \exp(-kh)\; q_n(y_{nk};\, b_n,\, S_h) \end{aligned}$$

and

$$I_n \leq \sum_{k=1}^{\infty} \exp(-(k-1)h)\; P\left(-\frac{h}{2b_n} \leq Y_n - y_{nk} < \frac{h}{2b_n}\right)$$

(2-34)
$$= \frac{h}{b_n} \sum_{k=1}^{k_h} \exp(-(k-1)h)\; q_n(y_{nk}: b_n, S_h)$$
$$+ \frac{h}{b_n} \sum_{k=k_h+1}^{\infty} \exp(-(k-1)h)\; q_n(y_{nk}: b_n, S_h).$$

Using (2-4) and (2-5) after noting that $y_{nk} \to 0$ as $n \to \infty$ for each k we get

(2-35)
$$\liminf_n (b_n I_n) \geq f(0)\; h \sum_{k=1}^{k_h} \exp(-kh)$$
$$= f(0)\; \frac{h(\exp(-h) - \exp(-(k_h+1)h))}{1-\exp(-h)}$$

and

(2-36)
$$\limsup_n (b_n I_n) \leq f(0)\; h \sum_{k=1}^{k_h} \exp(-(k-1)h) + M\,h \sum_{k=k_h+1}^{\infty} \exp(-(k-1)h)$$
$$= f(0)\; \frac{h(1-\exp(-k_h h))}{1-\exp(-h)} + M\,h\, \frac{\exp(-k_h h)}{1-\exp(-h)}.$$

Letting $h \to 0$ in (2-35) and (2-36) we get,

(2-37)
$$\lim_n (b_n I_n) = f(0).$$

This completes the proof of the lemma.◇

Corollary 2.7. Let $\{Y_n, n \geq 1\}$ be a sequence of random variables. Let $\{b_n\}$ be a sequence of real numbers such that $b_n \to \infty$. Assume that $\{Y_n, n \geq 1\}$ satisfies the conditions of any one of the Theorems 2.1, 2.2 or 2.3, then the conclusion (2-31) holds for the sequence $\{Y_n, n \geq 1\}$.

3. Strong Large Deviation Theorems

Let $\{T_n, n \geq 1\}$ be a sequence of random variables. Let $\{a_n\}$ be a sequence of real numbers such that $a_n \to \infty$ and $\{m_n\}$ be a bounded sequence of real numbers. Weak large deviation results give asymptotic expressions for $\log P(T_n/a_n \geq m_n)$ where the event $\{T_n/a_n \geq m_n\}$ represents a large deviation. A number of authors, including Seivers(1969), Steinebach(1978), Ellis(1984) have obtained such results under suitable conditions on the m.g.f. of T_n. Strong large deviation results give asymptotic expressions for $P(T_n/a_n \geq m_n)$. One of the earliest strong large deviation theorem was obtained by Bahadur and Ranga Rao(1960) when T_n is the sum of i.i.d. random variables. In Theorems 3.1 and 3.5 of this section we obtain strong large deviation limit theorems for arbitrary sequences of random variables $\{T_n, n \geq 1\}$, under some conditions on the m.g.f.'s of T_n's.

In Remarks 3.4 and 4.10 we show that the original result of Bahadur and Ranga Rao(1960) for sums of i.i.d. random variables follows as a corollary to our main Theorem 3.1.

The proofs of our strong large deviation results depend heavily on the local limit theorems of Section 2. We use the notation $A_n \sim B_n$, if $A_n/B_n \to 1$. We shall develop some more notation before stating the main theorem.

Let $\{T_n, n \geq 1\}$ be a sequence of random variables with m.g.f. $\phi_n(z) = E[\exp(zT_n)]$, which is nonvanishing and analytic in the region $\Omega = \{z \in \mathcal{C} : |z| < a\}$, where $a > 0$ and $\mathcal{C}$ is the set of all complex numbers. Let $\{a_n\}$ be a sequence of real numbers. Let

$$\psi_n(z) = a_n^{-1} \log \phi_n(z), \qquad \text{for } z \in \Omega, \text{ and} \tag{3-1}$$

$$\gamma_n(u) = \sup_{|s|<a,\, s \in R_1} [us - \psi_n(s)], \quad \text{for } u \in R_1. \tag{3-2}$$

Theorem 3.1. Let $\{T_n, n \geq 1\}$ be a sequence of random variables. Let $\{m_n\}$ be a bounded sequence of real numbers such that there exists a sequence $\{\tau_n\}$ satisfying $\psi_n'(\tau_n) = m_n$

and $0 < d < \tau_n < a_0 < a$ for some positive numbers a_0, d and for all $n \geq 1$. Let $a_n \to \infty$. Assume the following conditions for T_n:

(A) There exists $\beta < \infty$ such that $|\psi_n(z)| < \beta$ for all $n \geq 1$, $z \in \Omega$.

(B) There exists $\delta_0 > 0$ such that

$$\sup_{\delta \leq |t| \leq \lambda} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right| = o\left(\frac{1}{\sqrt{a_n}}\right)$$

for all $0 < \delta < \delta_0$ and for each $\lambda > \delta_0$.

(C) There exists $\alpha > 0$ such that $\psi_n''(\tau_n) \geq \alpha$ for all $n \geq 1$.

Then

$$P\left(\frac{T_n}{a_n} \geq m_n\right) \sim \frac{1}{\tau_n \sqrt{2\pi\, a_n \psi_n''(\tau_n)}} \exp(-a_n \gamma_n(m_n)). \tag{3-3}$$

Proof. Let K_n be the d.f. of T_n. We will use the left continuous version of the distribution function which will enable us to write the identities in (3-5). Let

$$H_n(y) = \int_{-\infty < u < y} \exp(u\tau_n - a_n \psi_n(\tau_n))\, dK_n(u), \tag{3-4}$$

and let T_n^* be a random variable with d.f. $H_n(y)$. Let $T_n' = T_n^* - a_n m_n$, $Y_n = T_n'/d_n$, $d_n = \sqrt{a_n \psi_n''(\tau_n)}$ and $b_n = \tau_n d_n$. Using these new random variables and the relation $\gamma_n(m_n) = m_n \tau_n - \psi_n(\tau_n)$, we have

$$\begin{aligned} P\left(\frac{T_n}{a_n} \geq m_n\right) &= \int_{a_n m_n}^{\infty} dK_n(y) \\ &= \int_{a_n m_n}^{\infty} \exp(-y\tau_n + a_n \psi_n(\tau_n))\, dH_n(y) \\ &= \exp(a_n \psi_n(\tau_n))\, E\big(\exp(-\tau_n T_n^*)\, I(T_n^* \geq a_n m_n)\big) \\ &= \exp(-a_n \gamma_n(m_n))\, E\big(\exp(-\tau_n T_n') I(T_n' \geq 0)\big) \\ &= \exp(-a_n \gamma_n(m_n))\, E\big(\exp(-b_n Y_n) I(Y_n \geq 0)\big). \end{aligned} \tag{3-5}$$

This step, usually called the use of Esscher transformation, is the starting point of most investigations in large deviations. The rest of the proof is stated and proved separately as Lemma 3.3 below where it is shown that when the conditions (A), (B) and (C) are satisfied Y_n converges in distribution to the standard normal and

$$b_n E(\exp(-b_n Y_n) I(Y_n \geq 0)) \to \frac{1}{\sqrt{2\pi}} \tag{3-6}$$

The present proof follows by substituting (3-6) in (3-5).◊

Remark 3.2. The boundedness of τ_n below by $d > 0$ is satisfied for example if $\liminf_n [(m_n - E(T_n)/a_n)] > 0$. A strong large deviation result for T_n when $\tau_n \to 0$ is proved later in Theorem 3.5.

We now state and prove Lemma 3.3, which will complete the proof of Theorem 3.1.

Lemma 3.3. Let $\{T_n, n \geq 1\}$ be a sequence of random variables. Let $a_n \to \infty$ and $\{m_n, n \geq 1\}$ be a bounded sequence of real numbers such that there exists a sequence $\{\tau_n\}$ satisfying $\psi_n'(\tau_n) = m_n$ and $0 < d < \tau_n < a_0 < a$ for some positive numbers a_0, d and for all $n \geq 1$. Let $d_n = \sqrt{a_n \psi_n''(\tau_n)}$. Let the rand. variable Y_n be defined as in the proof of Theorem 3.1. Assume that the conditions (A), (B) and (C) of Theorem 3.1 are satisfied. Then Y_n converges in distribution to standard normal and (3-6) holds.

Proof. The c.f. of Y_n is given by

$$\hat{f}_n(t) = \exp(-it a_n m_n / d_n) \frac{\phi_n(\tau_n + it/d_n)}{\phi_n(\tau_n)}. \tag{3-7}$$

Since $\psi_n(z) = a_n^{-1} \log \phi_n(z)$ is a finite and analytic function in Ω, and $|\tau_n| < a_0$, using condition (A) and Cauchy's theorem for derivatives it follows that

$$|\psi_n^{(k)}(\tau_n + it)| \leq \frac{k!\,\beta}{(a - a_0)^k}, \quad \text{for } k \geq 1, \tag{3-8}$$

for $|t| < (a - a_0)/2$. Using the Taylor series expansion we can write

$$\psi_n(\tau_n + it) = \psi_n(\tau_n) + it\psi_n'(\tau_n) - (t^2/2)\psi_n''(\tau_n) + R_n(\tau_n + it), \tag{3-9}$$

for $|t| < (a - a_0)/2$, where the remainder term R_n satisfies

$$|R_n(\tau_n + it)| \leq \frac{2\beta|t|^3}{(a - a_0)^3}. \tag{3-10}$$

From (3-9), (3-10) and condition (C) we obtain, for any fixed t, that

(3-11)
$$\begin{aligned} \log \hat{f}_n(t) &= -(ita_nm_n)/d_n + a_n\left[\psi_n(\tau_n + it/d_n) - \psi_n(\tau_n)\right] \\ &= -(ita_nm_n)/d_n + a_n\left[it\psi_n'(\tau_n)/d_n - (t^2\psi_n''(\tau_n))/(2d_n^2) + R_n(\tau_n + it/d_n)\right] \\ &= -t^2/2 + a_nR_n(\tau_n + it/d_n), \end{aligned}$$

and

$$|a_nR_n(\tau_n + it/d_n)| \leq \frac{2\beta|t|^3}{\alpha d_n(a - a_0)^3} \to 0, \quad \text{as } n \to \infty. \tag{3-12}$$

Hence Y_n converges in distribution to the standard normal random variable. We now proceed to show that Y_n satisfies the conditions (2-3) and (2-21) of Theorem 2.3. Let

$$\begin{aligned} g_n(t) &= d_n^{-2} \log |\hat{f}_n(d_nt)| \\ &= \frac{1}{\psi_n''(\tau_n)} \left[\mathrm{Real}(\psi_n(\tau_n + it) - \psi_n(\tau_n))\right]. \end{aligned} \tag{3-13}$$

Thus

$$g_n''(t) = \frac{-\operatorname{Real}(\psi_n''(\tau_n + it))}{\psi_n''(\tau_n)}$$

(3-14)
$$= \frac{-\operatorname{Real}(\psi_n''(\tau_n) + it\xi_n)}{\psi_n''(\tau_n)}$$
$$= -1 + \operatorname{Real}(it\xi_n/\psi_n''(\tau_n))$$
$$\leq -1 + |t||\xi_n|/\alpha,$$

where ξ_n is an appropriate complex number depending on the third derivative of ψ_n, which from (3-8) satisfies

(3-15)
$$|\xi_n| \leq \frac{3!\,\beta}{(a - a_0)^3} \quad \text{for } n \geq 1.$$

Therefore we can find $\delta > 0$ such that $\delta < \delta_0$ and for $|t| < \delta$,

(3-16)
$$g_n''(t) \leq -(1/2) \qquad \text{for all } n \geq 1.$$

This verifies condition (2-27) of Theorem 2.5, and hence condition (2-3) with $\beta_n = \delta d_n$. Now, with $b_n = \tau_n d_n$ and using condition (B) for fixed $\lambda > 0$, we get that,

$$\sup_{\delta d_n \leq |t| \leq \lambda b_n} |\hat{f}_n(t)| = \sup_{\delta \leq |t| \leq \lambda \tau_n} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right|$$

(3-17)
$$= o\left(\frac{1}{\sqrt{a_n}}\right)$$
$$= o\left(\frac{1}{\tau_n d_n}\right)$$

since $d < \tau_n < a_0$ and $d_n = O(\sqrt{a_n})$. This verifies condition (2-21) of Theorem 2.3 with $b_n = \tau_n d_n$. The assertion (3-6) now follows from Corollary 2.7. $\diamond$

Remark 3.4. Let $X_1, X_2, \ldots,$ be i.i.d. random variables with m.g.f. $\phi(z)$ and let $\psi(z) = \log(\phi(z))$, be finite for $|z| < a$. Let $T_n = X_1 + \cdots + X_n$. Bahadur and Ranga Rao (1960)

obtained a strong large deviation theorem for T_n in Theorem 1 of their paper considering three cases. In *Case* 1 they assumed that X_1 is absolutely continuous or more generally the d.f. satisfies Cramér's condition. In *Case* 2 they assumed that X_1 is a lattice variable and finally they consider all the other possibilities in *Case* 3. We now show that *Cases* 1 and 3 of Theorem 1 of Bahadur and Ranga Rao(1960) follows from our Theorem 3.1. *Case* 2 follows from our Theorem 4.9 and is obtained later in Remark 4.10. Assume now that X_1 is a nonlattice random variable. The m.g.f. of T_n is given by $\phi_n(z) = \phi^n(z)$. Let m be a real number such that there exists $0 < \tau < a$ satisfying $\psi'(\tau) = m$. We will now proceed to verify the conditions of our main Theorem 3.1, with $m_n = m$ and $a_n = n$ for all $n \geq 1$. Conditions (A) and (C) are trivially satisfied since $\psi_n \equiv \psi$ and $\tau_n = \tau$ for all $n \geq 1$. Using the fact that X_1 is nonlattice we get that for each $\delta > 0$ and $\lambda > \delta$ there exists $0 < \epsilon < 1$ such that

$$\text{(3-18)} \qquad \sup_{\delta \leq |t| \leq \lambda} \left| \frac{\phi(\tau + it)}{\phi(\tau)} \right| < (1 - \epsilon).$$

This shows that condition (B) is satisfied since $\phi_n(z) = \phi^n(z)$. Thus the conclusion of Theorem 3.1 holds. This proves the strong large deviation result for $P(T_n/n \geq m)$ contained in *Cases* 1 and 3 of Theorem 1 of Bahadur and Ranga Rao(1960).

We now turn our attention to the case where $\tau_n \to 0$ as $n \to \infty$, but not very fast. More specifically, we require that $\tau_n \sqrt{a_n} \to \infty$. In this case we can get the stronger result that the conclusion of Theorem 3.1 holds without condition (B).

Theorem 3.5. Let $\{T_n,\ n \geq 1\}$ be a sequence of random variables. Let $\{m_n\}$ be a sequence of real numbers such that there exists a sequence $\{\tau_n\}$ satisfying $\psi_n'(\tau_n) = m_n$, and $\tau_n > 0$. Also assume that $\tau_n \to 0$ and $\tau_n \sqrt{a_n} \to \infty$. Let T_n satisfy the conditions (A) and (C) of Theorem 3.1. Then

$$\text{(3-19)} \qquad P\left(\frac{T_n}{a_n} \geq m_n\right) \sim \frac{1}{\tau_n \sqrt{2\pi\, a_n \psi_n''(\tau_n)}} \exp(-a_n \gamma_n(m_n)).$$

The proof of Theorem 3.5 is similar to the proof of Theorem 3.1. The only change is that we apply Lemma 3.6 instead of Lemma 3.3 to obtain (3-6).

Lemma 3.6. Let $\{Y_n, n \geq 1\}$ be a sequence of random variables as defined in the proof of Theorem 3.1. Let $\tau_n \to 0$ and $\tau_n\sqrt{a_n} \to \infty$. Assume that conditions (A) and (C) of Theorem 3.1 are satisfied. Then Y_n converges in distribution to standard normal and (3-6) holds.

Proof. We have already seen that in Lemma 3.3, Y_n converges in distribution to standard normal random variable if conditions (A) and (C) are satisfied. Also, Y_n satisfies condition (2-3) of Theorem 2.2 with $\beta_n = \delta d_n$. Let $b_n = \tau_n d_n$. The assumptions on τ_n imply that $b_n \to \infty$ and $\beta_n/b_n \to \infty$ as $n \to \infty$. Therefore the conclusions (2-4) and (2-5) are valid for Y_n by Theorem 2.2. Thus Y_n satifies (3-6) by Corollary 2.7. This proves Lemma 3.6. $\diamondsuit$

4. The Lattice Case

This section primarily deals with local limit theorems and strong large deviation theorems for lattice valued random variables. These theorems are analogous to the theorems of Sections 2 and 3. In this section and the next we use the word span to indicate the maximal span of a lattice random variable.

Theorem 4.1. Let Y_n be a lattice valued random variable taking values in the lattice $\{kh_n : k = 0, \pm 1, \pm 2, \ldots\}$, where $h_n > 0$ and $n \geq 1$. Assume that the span h_n of Y_n, converges to zero as $n \to \infty$. Let Y_n converge in distribution to Y. Let $\hat{f}_n$ be the c.f. of Y_n and $\hat{f}$ be the c.f. of Y. Assume that there exists an integrable function f^* such that

$$\sup_n |\hat{f}_n(t)|\, I(|t| \leq \pi/h_n) \leq f^*(t) \tag{4-1}$$

for each t. Then Y possesses a bounded and continuous p.d.f., f, and there exists a constant $M < \infty$ such that

$$\sup_n \left[\frac{1}{h_n} P(Y_n = y) \right] \leq M \tag{4-2}$$

uniformly in y. Further, if y_n is in the range of Y_n, and y_n converges to y^* then

$$\frac{1}{h_n} P(Y_n = y_n) \to f(y^*) \tag{4-3}$$

as $n \to \infty$.

Proof. Let y_n be in the range of Y_n. Then an application of the inversion formula yields

$$\frac{1}{h_n} P(Y_n = y_n) = \frac{1}{2\pi} \int_{-\pi/h_n}^{\pi/h_n} \exp(-ity_n)\, \hat{f}_n(t)\, dt \tag{4-4}$$

The assertions (4-2) and (4-3) now follow from condition (4-1) and the dominated convergence theorem.

The next theorem relaxes condition (4-1) but imposes an additional condition (4-6) and obtains the same conclusion as Theorem 4.1.

Theorem 4.2. Let $\{Y_n, n \geq 1\}$ be a sequence of lattice valued random variables as in Theorem 4.1. Assume that there exists an integrable function f^* such that

$$\sup_n |\hat{f}_n(t)| \, I(|t| < \beta_n) \leq f^*(t) \tag{4-5}$$

for each t, and

$$\theta_n^* \stackrel{\text{def}}{=} \sup_{\beta_n \leq |t| \leq \pi/h_n} |\hat{f}_n(t)| = o(h_n), \tag{4-6}$$

as $n \to \infty$, for some sequence of real numbers $\{\beta_n\}$ such that $\beta_n \to \infty$ and $\beta_n < \pi/h_n$ for all $n \geq 1$. Then Y possesses a bounded and continuous p.d.f., f, and there exists constants M such that (4-2) holds uniformly for all y. If y_n be in the range of Y_n and y_n converges to y^* as $n \to \infty$ then (4-3) holds.

Proof. Let y_n is a possible value of Y_n. Then an application of the inversion formula yields

(4-7)
$$\begin{aligned}
\frac{1}{h_n} P(Y_n = y_n) &= \frac{1}{2\pi} \int_{-\pi/h_n}^{\pi/h_n} \exp(-ity_n) \, \hat{f}_n(t) \, dt \\
&= \frac{1}{2\pi} \int_{|t| < \beta_n} \exp(-ity_n) \, \hat{f}_n(t) \, dt + \frac{1}{2\pi} \int_{\beta_n \leq |t| \leq \pi/h_n} \exp(-ity_n) \, \hat{f}_n(t) \, dt \\
&= I_{n1} + I_{n2} \qquad \text{(say)}.
\end{aligned}$$

It is easy to check that condition (4-5) and dominated convergence theorem imply that I_{n1} converges to $f(y^*) = (1/2\pi) \int \exp(-ity^*) \, \hat{f}(t) \, dt$. Next

$$\begin{aligned}
|I_{n2}| &\leq \frac{1}{h_n} \sup_{\beta_n \leq |t| \leq \pi/h_n} |\hat{f}_n(t)| \\
&= \frac{\theta_n^*}{h_n}
\end{aligned} \tag{4-8}$$

which converges to zero as $n \to \infty$, by condition (4-6). This completes the proof of (4-3).

Next, from (4-7) and (4-8) we get

$$\left|\frac{1}{h_n} P(Y_n = y)\right| \le \frac{1}{2\pi} \int_{|t|<\beta_n} |\hat{f}_n(t)|\, dt + \frac{\theta_n^*}{h_n}$$

$$\text{(4-9)} \qquad \le \frac{1}{2\pi} \int_{-\infty}^{\infty} f^*(t)\, dt + \frac{\theta_n^*}{h_n}$$

$$\le \frac{1}{\pi} \int_{-\infty}^{\infty} f^*(t)\, dt = M$$

uniformly in y, for $n \ge n_0$. This completes the proof of the theorem. $\diamond$

Theorem 4.3. Let Y_n be a lattice valued random variable taking values in the lattice $\{kh_n : k = 0, \pm 1, \pm 2, \ldots\}$ where $h_n > 0$ for $n \ge 1$. Assume that the span h_n of Y_n converges to zero as $n \to \infty$. Let Y_n converge in distribution to Y. Let $\{b_n\}$ be a sequence of real numbers such that $\lim_n (b_n h_n) = \infty$. Suppose that Y_n satisfies the conditions (4-2) and (4-3). Then

$$\text{(4-10)} \qquad \frac{1}{h_n} E\left[\exp(-b_n Y_n)\, I(Y_n \ge 0)\right] \to f(0)$$

as $n \to \infty$, where f is the p.d.f. of Y.

Proof. Consider

$$I_n = E\big(\exp(-b_n Y_n) I(Y_n \ge 0)\big)$$

$$\text{(4-11)} \qquad = P(Y_n = 0) + \sum_{k=1}^{\infty} \exp(-k b_n h_n) P\big(Y_n = k h_n\big).$$

Using (4-2) we get

$$\text{(4-12)} \qquad \left[\frac{1}{h_n} \sum_{k=1}^{\infty} \exp(-k b_n h_n) P\big(Y_n = k h_n\big)\right] \le M \frac{\exp(-b_n h_n)}{(1 - \exp(-b_n h_n))} \to 0$$

as $n \to \infty$, since $\lim(b_n h_n) = \infty$. Therefore from (4-11), (4-12) and using (4-3) we get

$$\text{(4-13)} \qquad \lim_n \left(\frac{1}{h_n} I_n\right) = \lim_n \frac{1}{h_n} P\big(Y_n = 0\big) = f(0).$$

This completes the proof of the theorem. $\diamond$

Theorem 4.4. Let Y_n be a lattice valued random variable taking values in the lattice $\{kh_n : k = 0, \pm 1, \pm 2, \ldots\}$ where $h_n > 0$ for $n \geq 1$. Assume that the span h_n of Y_n converges to zero as $n \to \infty$. Let Y_n converge in distribution to Y. Let $\{b_n\}$ be a sequence of real numbers such that $0 < \liminf_n(b_n h_n) = b < \infty$. Suppose that Y_n satisfies the conditions (4-2) and (4-3). Then

$$\text{(4-14)} \qquad \frac{(1 - \exp(-b_n h_n))}{h_n} E[\exp(-b_n Y_n) I(Y_n \geq 0)] \to f(0)$$

as $n \to \infty$, where f is the p.d.f. of Y.

Proof. Consider

$$\text{(4-15)} \qquad \begin{aligned} I_n &= E(\exp(-b_n Y_n) I(Y_n \geq 0)) \\ &= \sum_{k=0}^{\infty} \exp(-kb_n h_n) P(Y_n = kh_n). \end{aligned}$$

Let $N > 1$ be fixed. A lower bound for I_n is given by

$$\text{(4-16)} \qquad \sum_{k=0}^{N-1} \exp(-kb_n h_n)\, P(Y_n = kh_n)$$

and an upper bound is given by

$$\text{(4-17)} \qquad \sum_{k=0}^{N-1} \exp(-kb_n h_n) P(Y_n = kh_n) + Mh_n \sum_{k=N}^{\infty} \exp(-kb_n h_n)$$

wherein we have used (4-2). Combining (4-15), (4-16), (4-17) and using (4-3) we get

$$\text{(4-18)} \qquad \begin{aligned} \liminf_n \left[\frac{(1 - \exp(-b_n h_n))}{h_n}\, I_n\right] &\geq f(0) \liminf_n (1 - \exp(-Nb_n h_n)) \\ &= f(0)\,(1 - \exp(-Nb)) \end{aligned}$$

and

$$\text{(4-19)} \qquad \begin{aligned} \limsup_n \left[\frac{(1 - \exp(-b_n h_n))}{h_n}\, I_n\right] &\leq f(0) + \limsup_n (M \exp(-Nb_n h_n)) \\ &= f(0) + M \exp(-Nb) \end{aligned}$$

where $b = \liminf_n(b_n h_n)$. Now letting $N \to \infty$ in (4-18) and (4-19) we get

$$\text{(4-20)} \qquad \lim_n \left[\frac{(1 - \exp(-b_n h_n))}{h_n}\, I_n\right] = f(0).$$

This completes the proof of the theorem. ◇

We are now in a position to state strong large deviation theorems for arbitrary sequence $\{T_n, n \geq 1\}$ of lattice random variables. We shall develop some notation before stating the theorems.

Let T_n be a lattice random variable taking values in the lattice $\{t_n + kp_n : k = 0, \pm 1, \pm 2, \ldots\}$, where $p_n > 0$ for $n \geq 1$. Let the m.g.f. of T_n, $\phi_n(z)$, be analytic and nonvanishing in the region $\Omega = \{z \in C : |z| < a\}$. Let $\{a_n\}$ be a sequence of real numbers such that $a_n \to \infty$. Let

$$\psi_n(z) = a_n^{-1} \log \phi_n(z). \tag{4-21}$$

be finite and analytic in Ω. For each n, let m_n be in the range of T_n/a_n and let τ_n be such that $\psi_n'(\tau_n) = m_n$ with $0 < \tau_n < a_0 < a$. Let

$$\gamma_n(m_n) = m_n\tau_n - \psi_n(\tau_n).$$

The following Theorems 4.5, 4.7, 4.8, 4.9 and 4.13 obtain an estimate of the large deviation probability, $P(T_n/a_n \geq m_n)$, depending on the various behaviors of p_n and τ_n.

Theorem 4.5. Assume that p_n and τ_n are such that $p_n \to \infty$, $p_n = o(\sqrt{a_n})$ and $\lim_n(\tau_n p_n) = \infty$. Assume that $\{T_n, n \geq 1\}$ satisfies conditions (A) and (C) of Theorem 3.1. Then

$$P\left(\frac{T_n}{a_n} \geq m_n\right) \sim \frac{p_n}{\sqrt{2\pi\, a_n \psi_n''(\tau_n)}} \exp(-a_n\gamma_n(m_n)). \tag{4-22}$$

Proof. Since m_n is in the range of T_n/a_n, we can write $a_n m_n = t_n + l_n p_n$ for some integer l_n. Using the relation $\gamma_n(m_n) = m_n\tau_n - \psi_n(\tau_n)$ we get

$$\begin{aligned} P\left(\frac{T_n}{a_n} \geq m_n\right) &= P(T_n \geq t_n + l_n p_n) \\ &= \sum_{k=0}^{\infty} P(T_n = t_n + (k + l_n)p_n) \\ &= \exp(-a_n\gamma_n(m_n)) \sum_{k=0}^{\infty} \exp(a_n\gamma_n(m_n)) P(T_n = t_n + (k + l_n)p_n) \\ &= \exp(-a_n\gamma_n(m_n)) \sum_{k=0}^{\infty} \exp(-k\tau_n p_n) P_n(k) \end{aligned} \tag{4-23}$$

where

$$P_n(k) = \frac{\exp((t_n + (k+l_n)p_n)\tau_n)}{\phi_n(\tau_n)} P(T_n = t_n + (k+l_n)p_n). \tag{4-24}$$

Let us introduce a lattice valued random variable T_n' which takes the value kp_n with probability $P_n(k)$ for each n. Therefore, we can rewrite (4-23) as

$$\begin{aligned} P\left(\frac{T_n}{a_n} \geq m_n\right) &= \exp(-a_n\gamma_n(m_n))\, E(\exp(-\tau_n T_n')\, I(T_n' \geq 0)) \\ &= \exp(-a_n\gamma_n(m_n))\, E(\exp(-\tau_n \beta_n Y_n)\, I(Y_n \geq 0)) \\ &= \exp(-a_n\gamma_n(m_n))\, E(\exp(-b_n Y_n)\, I(Y_n \geq 0)) \\ &= \exp(-a_n\gamma_n(m_n))\, I_n, \end{aligned} \tag{4-25}$$

where $d_n = \sqrt{a_n\psi''_n(\tau_n)}$ and $Y_n = T_n'/d_n$ and $b_n = \tau_n d_n$. Note that Y_n is a lattice valued random variable with span $h_n = p_n/d_n$ and $\lim_n(b_n h_n) = \lim_n(\tau_n p_n) = \infty$. If the conditions (A) and (C) are satisfied, the next Lemma 4.6 shows that Y_n converges in distribution to the standard normal and satisfies (4-2) and (4-3). Therefore applying Theorem 4.3 for Y_n we get

$$I_n \sim \frac{h_n}{\sqrt{2\pi}} \tag{4-26}$$

as $n \to \infty$. Substituting (4-26) in (4-25) we get (4-22). This completes the proof of the theorem. $\diamond$

We now state and prove Lemma 4.6 which was used in a major way in the proof of the above theorem.

Lemma 4.6. Let the lattice random variable Y_n be defined as in the proof of Theorem 4.5. Note that Y_n takes values in the lattice $\{kh_n : k = 0, \pm1, \pm2, \ldots\}$, with probabilities $\{P_n(k) : k = 0, \pm1, \pm2, \ldots\}$, where $h_n = p_n/d_n$ and $P_n(k)$ is as defined in (4-24). Assume that T_n satisfies conditions (A) and (C) of Theorem 3.1. Let $p_n \to \infty$ such that $p_n = o(\sqrt{a_n})$. Then Y_n converges in distribution to standard normal and satisfies (4-2) and (4-3).

Proof. The lemma will be proved once we verify that Y_n satisfies the conditions of Theorem 4.1. Note that $h_n \to 0$, since $p_n = o(\sqrt{a_n})$ and $\psi_n''(\tau_n) > \alpha$. The c.f. of Y_n is given by

$$
\begin{aligned}
\hat{f}_n(t) &= E(\exp(itY_n)) \\
&= \sum_{k=-\infty}^{\infty} \exp(itkh_n) P_n(k) \\
&= \sum_{k=-\infty}^{\infty} \exp(it(k-l_n)h_n + (t_n + kp_n)\tau_n) \frac{P(T_n = t_n + kp_n)}{\phi_n(\tau_n)} \qquad (4\text{-}27) \\
&= \exp(-ita_n m_n/d_n) \frac{\phi_n(\tau_n + it/d_n)}{\phi_n(\tau_n)},
\end{aligned}
$$

wherein we have used the fact $a_n m_n = t_n + l_n p_n$. If the conditions (A) and (C) are satisfied Lemma 3.3 shows that $\hat{f}_n(t)$ converges to $\exp(-t^2/2)$ and there exists $\delta > 0$ such that

$$
|\hat{f}_n(t)|\, I(|t| < \delta d_n) \le \exp(-\alpha t^2/2) \qquad (4\text{-}28)
$$

for $n \ge 1$. Since $p_n \to \infty$, we can choose n large such that $p_n > \pi/\delta$ and hence $\pi/h_n < \delta d_n$. Thus for sufficiently large n we have

$$
|\hat{f}_n(t)|\, I(|t| \le \pi/h_n) \le f^*(t) = \exp(-\alpha t^2/2) \qquad (4\text{-}29)
$$

This verifies (4-1). The lemma now follows from Theorem 4.1. $\diamond$

Theorem 4.7. Assume that p_n and τ_n are such that $p_n \to \infty$, $p_n = o(\sqrt{a_n})$ and $\liminf_n(\tau_n p_n) = b < \infty$. Let $\{T_n,\, n \ge 1\}$ satisfies conditions (A) and (C) of Theorem 3.1. Then

$$
P\left(\frac{T_n}{a_n} \ge m_n\right) \sim \frac{p_n}{\sqrt{2\pi\, a_n \psi_n''(\tau_n)}} \frac{\exp(-a_n\gamma_n(m_n))}{(1-\exp(-\tau_n p_n))}. \qquad (4\text{-}30)
$$

Proof. Proceeding as in the proof of Theorem 4.5 we get that

$$
\begin{aligned}
P\left(\frac{T_n}{a_n} \ge m_n\right) &= \exp(-a_n\gamma_n(m_n))\, E\big(\exp(-\tau_n d_n Y_n) I(Y_n \ge 0)\big) \\
&= \exp(-a_n\gamma_n(m_n))\, E\big(\exp(-b_n Y_n) I(Y_n \ge 0)\big) \qquad (4\text{-}31) \\
&= \exp(-a_n\gamma_n(m_n))\, I_n
\end{aligned}
$$

where Y_n is as defined in the proof of Theorem 4.5 and $b_n = \tau_n d_n$. Note that $\liminf_n(b_n h_n) = \liminf_n(\tau_n p_n) = b < \infty$. The theorem now follows from Lemma 4.6 and Theorem 4.4. $\diamond$

Theorem 4.8. Assume that p_n and τ_n are such that $p_n \to 0$ and $0 < d < \tau_n < a_0 < a$. Suppose that T_n satisfy conditions (A), (B) and (C) of Theorem 3.1. Then

$$P\left(\frac{T_n}{a_n} \geq m_n\right) \sim \frac{1}{\tau_n \sqrt{2\pi\, a_n \psi_n''(\tau_n)}} \exp(-a_n \gamma_n(m_n)). \tag{4-32}$$

Proof. Proceeding as in the proof of Theorem 4.5 we get that

$$P\left(\frac{T_n}{a_n} \geq m_n\right) = \exp(-a_n \gamma_n(m_n))\, E(\exp(-b_n Y_n) I(Y_n \geq 0)) \tag{4-33}$$

where Y_n is as defined in the proof of Theorem 4.5 and $b_n = \tau_n d_n$. The rest of the proof is similar to the proof of Theorem 3.1. $\diamond$

Theorem 4.9. Assume that there exists positive numbers p^*, p^{**} and d such that $p^* < p_n < p^{**}$ and $\tau_n > d$ for all $n \geq 1$. Let T_n satisfy conditions (A), (C) of Theorem 3.1 and the following condition (B′):

(B′) There exists $\delta_1 > 0$, such that for $0 < \delta < \delta_1$,

$$\sup_{\delta \leq |t| \leq \pi/p_n} \left|\frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)}\right| = o\left[\frac{1}{\sqrt{a_n}}\right].$$

Then

$$P\left(\frac{T_n}{a_n} \geq m_n\right) \sim \frac{p_n}{\sqrt{2\pi\, a_n \psi_n''(\tau_n)}} \frac{\exp(-a_n \gamma_n(m_n))}{(1 - \exp(-\tau_n p_n))}. \tag{4-34}$$

Proof. The proof of this theorem is similar to the proof of Theorem 4.7. The major difference is that we apply Lemma 4.12 instead of Lemma 4.6. If the conditions (A), (B′) and (C) are satisfied, the next Lemma 4.12 shows that Y_n converges in distribution to the standard normal and it satisfies (4-2) and (4-3). Note that in this case $\liminf_n(b_n h_n) =$

$\liminf_n(\tau_n p_n) = h < \infty$. Combining Lemma 4.12 and Theorem 4.4 we get the conclusion (4-34).◇

Remark 4.10. Let $\{T_n, n \geq 1\}$ be a sequence of lattice random variables with span $p_n > 0$. If $p_n \to 0$ then Condition (B′) implies Condition (B). Therefore one should note that Theorem 4.8 is also *true* if Condition (B) is replaced by Condition(B′). However, if p_n is such that $0 < p^* < p_n < p^{**} < \infty$ for some positive constants p^* and p^{**}, the stronger Condition (B′) is never satisfied for the lattice random variables $\{T_n, n \geq 1\}$. In this case Theorem 4.9 obtains the strong large deviation result assuming the weaker condition (B′). In the case where $p_n \to \infty$, both conditions (B) and (B′) can be dropped altogether as shown in Theorems 4.5, 4.7 and 4.13.

Remark 4.11. We now show that *Case* 2 of Theorem 1 of Bahadur and Ranga Rao(1960) follows from Theorem 4.9. Let $X_1, X_2, \ldots,$ be i.i.d. lattice random variables with span $p > 0$ and m.g.f. given by $\phi(z)$. Let $\psi(z) = \log(\phi(z))$, be finite for $|z| < a$. Let $T_n = X_1 + \cdots + X_n$. The m.g.f. of T_n is given by $\phi_n(z) = \phi^n(z)$. Let m be a real number such that nm is a possible value of T_n and there exists $0 < \tau < a$ satisfying $\psi'(\tau) = m$. Conditions (A) and (C) are trivially satisfied as noted in Remark 3.4. Using the fact that X_1 is lattice with span p we get that for each $\delta > 0$ there exists $0 < \epsilon < 1$ such that

$$\sup_{\delta \leq |t| \leq \pi/p} \left| \frac{\phi(\tau + it)}{\phi(\tau)} \right| < (1 - \epsilon). \tag{4-35}$$

This shows that condition (B′) is satisfied since $\phi_n(z) = \phi^n(z)$. Thus the conclusion of Theorem 4.9 holds. This proves the strong large deviation result for $P(T_n/n \geq m)$ contained in *Case* 2 of Theorem 1 of Bahadur and Ranga Rao(1960) and also Theorem 4 of Blackwell and Hodges(1959).

Lemma 4.12. Let Y_n be a function of the lattice valued random variable T_n as in Lemma 4.6. Let there exists positive numbers p^*, p^{**} and d such that $p^* < p_n < p^{**}$ and $\tau > d$ for all $n \geq 1$. If T_n satisfies conditions (A), (B′) and (C) then Y_n converges in distribution to standard normal and satisfies (4-2) and (4-3).

Proof. The lemma follows once we verify that Y_n satisfies the conditions of Theorem 4.2. Note that the span $h_n = p_n/d_n$ of Y_n goes to zero as $n \to \infty$, since $d_n \to \infty$ and p_n is bounded. If T_n satisfies conditions (A) and (C) then Lemma 4.6 shows that Y_n converges to standard normal and satisfies (4-5) with $\beta_n = \delta d_n$. Further, condition (B′) implies that Y_n satisfies (4-6). Thus Y_n satisfies all the conditions of Theorem 4.2.◇

Theorem 4.13. Let T_n satisfy conditions (A), (C) of Theorem 3.1. Assume that the span p_n of T_n and τ_n satisfy any one of the following conditions:

(i) $p_n \to \infty$, $\tau_n p_n \to 0$, $\tau_n \sqrt{a_n} \to \infty$.

(ii) $0 < p_n < p < \infty$, $\tau_n \to 0$, $\tau_n \sqrt{a_n} \to \infty$.

Then

$$P\left(\frac{T_n}{a_n} \geq m_n\right) \sim \frac{1}{\tau_n \sqrt{2\pi\, a_n \psi_n''(\tau_n)}} \exp(-a_n \gamma_n(m_n)). \tag{4-36}$$

Proof. Proceeding as in Theorem 4.5 we can write

$$P\left(\frac{T_n}{a_n} \geq m_n\right) = \exp(-a_n \gamma_n(m_n))\, E\big(\exp(-b_n Y_n) I(Y_n \geq 0)\big) \tag{4-37}$$

where Y_n is as defined in the proof of Theorem 4.5 and $b_n = \tau_n d_n$. The rest of the proof is similar to the proof of Theorem 3.5.◇

5. Applications

In this section we give two typical applications to illustrate the large deviation limit theorems and strong large deviation limit theorems of the previous sections. The first example is a local limit result and illustrates Theorem 2.1. The second example is a strong large deviation result for a lattice valued random variable and illustrates the theorems in Section 4.

Example 5.1. This example applies to a general class of sums of dependent random variables considered in Chaganty and Sethuraman(1987). Though it was proved in that paper that the limit distribution could be both normal and nonnormal, our example applies only to the case where the limit distribution is normal. We first present a particular application and then state a more general application referring to conditions found in Chaganty and Sethuraman(1987).

Let $\{X_1^{(n)}, X_2^{(n)}, \ldots, X_n^{(n)}\}$ be a triangular array of random variables with joint density function

$$(5\text{-}1) \qquad dQ_n^*(\mathbf{x}) = z_n^{-1}(2\pi)^{-n/2}\left[\cosh\left(\frac{s_n}{\beta n}\right)\right]^n \exp\left(-\sum_{j=1}^{n}\frac{x_j^2}{2}\right)\, d\mathbf{x},$$

where $\mathbf{x} = (x_1, \ldots, x_n)$, $s_n = x_1 + \ldots + x_n$, $\beta > 1$ and z_n is a normalizing constant. Such dependent random variables arise in generalized Curie-Weiss models used to describe ferro-magnets. Using Theorem 3.7 of Chaganty and Sethuraman(1987) or using (5-3) below we can show that $Y_n = (X_1^{(n)} + \ldots + X_n^{(n)})/\sqrt{n}$ converges in distribution to a normal distribution with mean 0 and variance $\sigma^2 = \beta^2/(\beta^2 - 1)$ (Example 4.4 of Chaganty and Sethuraman(1987) considered the case $\beta = 1$ and obtained a non-normal distribution under a different normalization). We will now show that Theorem 2.1 applies to Y_n. Since

$$(5\text{-}2) \qquad (\cosh \omega)^n = \sum_{y \in C_n} \exp(\omega y)\lambda_n(y)$$

with $\lambda_n(y) = \binom{n}{(n+y)/2} 2^{-n}$ and $C_n = \{-n, -n+2, \ldots, n\}$, the c.f. of Y_n is given by

$$
\begin{aligned}
(5\text{-}3) \qquad \hat{f}_n(t) &= E(\exp(itY_n)) \\
&= z_n^{-1} \sum_{y \in C_n} \left[\frac{1}{(2\pi)^{n/2}} \int \exp\left(\frac{its_n}{\sqrt{n}} + \frac{ys_n}{\beta n} - \sum_{j=1}^{n} \frac{x_j^2}{2} \right) d\mathbf{x} \right] \lambda_n(y) \\
&= \exp(-t^2/2) z_n^{-1} \sum_{y \in C_n} \exp\left(\frac{ity}{\beta\sqrt{n}} + \frac{y^2}{2\beta^2 n} \right) \lambda_n(y).
\end{aligned}
$$

Since $\hat{f}_n(0) = 1$, we have

$$
(5\text{-}4) \qquad |\hat{f}_n(t)| \le \exp(-t^2/2) \qquad \text{for all } n \text{ and } t.
$$

Thus from Theorem 2.1 it follows that for any $h > 0$, $\{b_n\} \to \infty$ and $y_n \to y$,

$$
(5\text{-}5) \qquad b_n P(|Y_n - y_n| < h/b_n) \to \frac{2h}{\sqrt{2\pi}\sigma} \exp\left(-\frac{y^2}{2\sigma^2} \right)
$$

with $\sigma^2 = \beta^2/(\beta^2 - 1)$.

From the above discussion and from a full use of Theorem 3.7 of Chaganty and Sethuraman(1987) we have the following application which we state without proof.

Let $\{X_1^{(n)}, X_2^{(n)}, \ldots, X_n^{(n)}\}$ be a triangular array of random variables whose joint distribution is as given in (3.13) of Theorem 3.7 of Chaganty and Sethuraman(1987). We will impose conditions on the probability measure P and the index r appearing in that Theorem. Let P be the standard normal distribution and let $r = 1$. Under these conditions, Theorem 3.7 of Chaganty and Sethuraman(1987) shows that there is a sequence of constants $\{m_n\}$ such that

$$
(5\text{-}6) \qquad Y_n = \Big(\sum_{j=1}^{n} X_i^{(n)} - nm_n\Big)/\sqrt{n}
$$

has a limiting normal distribution with mean 0 and variance σ^2. Let $\hat{f}_n(t)$ be the c.f. of Y_n. For this case, if we proceed as in the application above, we can establish (5-4) for all n and t. This shows that (5-5) is true with the appropriate σ.

Example 5.2. We now obtain a strong large deviation result for the Wilcoxon signed-rank statistic under the null hypothesis. This stregthens the well known weak large deviation results for this statistic (see Klotz(1965)).

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. continuous random variables with median m. Arrange $|X_1|, |X_2|, \ldots, |X_n|$ in increasing order of magnitude and assign ranks $1, 2, \ldots, n$. The Wilcoxon signed-rank statistic U_n is defined as the sum of the ranks of positive X_i's. The statistic U_n is used to test the null hypothesis $H_0 : m = 0$ vs $H_1 : m \neq 0$. Let $T_n = U_n/n$. The random variable T_n is a lattice random variable with span $p_n = 1/n$. The m.g.f. of T_n under the null hypothesis H_0 is given by

$$\phi_n(z) = \prod_{k=1}^{n} \left[(\exp(kz/n) + 1)/2 \right], \qquad z \in C. \tag{5-7}$$

It is easy to check that $\phi_n(z)$ is analytic and nonvanishing in the region $\Omega = \{z \in C : |z| < \pi/2\}$. Let

$$\psi_n(z) = n^{-1} \log \phi_n(z). \tag{5-8}$$

It is easy to check that there exists $\beta > 0$ such that $|\psi_n(z)| < \beta$ for $|z| < \pi/2$. Straightforward calculations show that $\psi_n''(\tau)$ is bounded below by a positive number α for real τ such that $|\tau| < \pi/2$. Thus T_n satisfies conditions (A) and (C). Next we first note that $\psi_n'(s) \to \int_0^1 (x)/(1 + \exp(-sx))\, dx$ and that the range of $\psi_n'(s)$, for real s contains the open interval (0, 1/2) for all $n \geq 1$. Thus if $\{m_n\}$ is a sequence of real numbers such that $1/4 < m_n < \overline{m} < \int_0^1 (x)/(1 + \exp(-\pi x/2))\, dx$ then we can find a positive number a_0 and a sequence $\{\tau_n\}$ satisfying $0 < \tau_n < a_0 < \pi/2$ and $\psi_n'(\tau_n) = m_n$, for all $n \geq 1$. If $\tau_n \to 0$ such that $\sqrt{n}\tau_n \to \infty$, then Theorem 4.13 shows that (4-36) is valid for $P(T_n \geq nm_n)$. Now consider the case where $\tau_n > d$ for all $n \geq 1$ for some positive d. From the analysis in Example 3.1 of Chaganty and Sethuraman(1985) it can be seen that there exists n_0 and $\delta_1 > 0$ such that for $0 < \delta < \delta_1$,

$$\sup_{\delta \leq |t| \leq \pi/p_n} \left| \frac{\phi_n(\tau_n + it)}{\phi_n(\tau_n)} \right| \leq \exp(-n\alpha\delta^2/4) \tag{5-9}$$

for $n \geq n_0$. Since $p_n \to 0$ this verifies condition (B). Therefore Theorem 4.8 shows that the conclusion (4-36) holds even in this case.

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An event is loosely called a large deviation event if the dominant term in the probability of that event goes to zero exponentially. Most papers give asymptotic expressions only to the logarithm of the probability of a large deviation event. In this paper we obtain strong deviation results which give asymptotic expressions to the actual probability of a large deviation event based on an arbitrary sequence of random variables, under some conditions on the moment generating functions. (please see back to continue)

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The proof of these results depends on local limit theorems, which also are proved in this paper, by imposing some conditions on the characteristic functions. A local limit theorem states that the pseudo-densities of random variables converge, which is stronger than the convergence of these random variables in distribution.